Generalized Micro-Structures for Non-Binary CSP*

Achref El Mouelhi

LSIS - UMR CNRS 7296
Aix-Marseille Université
Avenue Escadrille Normandie-Niemen
13397 Marseille Cedex 20 (France)
achref.elmouelhi@lsis.org

Abstract. In [6], a new approach has been introduced for improving the solving of binary CSPs. This approach is based on the notion of micro-structure for binary CSPs. This formalism was used to analyze the theoretical complexity of binary CSPs.
In this paper, we generalize the notion of micro-structure to non-binary CSPs in order to retain the tractability as in the binary case.

1 Preliminaries

Constraint Satisfaction Problems (CSPs [10]) provide an efficient way of formulating problems in computer science, especially in Artificial Intelligence such as scheduling, temporal reasoning, planning, graph problems, just to name a few.

Formally, a *constraint satisfaction problem* is a triple (X, D, C), where $X = \{x_1, \ldots, x_n\}$ is a set of variables, $D = (D_{x_1}, \ldots, D_{x_n})$ is a list of finite domains of values, one per variable, and $C = \{C_1, \ldots, C_e\}$ is a finite set of constraints. Each constraint C_i is a couple $(S(C_i), R(C_i))$, where $S(C_i) = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq X$ is the *scope* of c_i, and $R(C_i) \subseteq D_{x_{i_1}} \times \cdots \times D_{x_{i_k}}$ is its *relation*. The *arity* of C_i is $|S(C_i)|$. A CSP is called *binary* if all constraints are of arity 2 (we denote C_{ij} the binary constraint whose scope is $S(C_{ij}) = \{x_i, x_j\}$). Otherwise, a CSP is said to be *n-ary*. A *consistent assignment* is an assignment that does not violate the constraints. A *solution* is a complete assignment that satisfies all constraints. Testing whether a CSP has a solution is known to be NP-complete.

We can associate to a binary CSP a graph called a micro-structure which is defined as follows:

Definition 1 (micro-structure [6]) *Given a binary CSP $P = (X, D, C)$, the* micro-structure *of P is the undirected graph $\mu(P) = (V, E)$ with:*

- $V = \{(x_i, v_i) : x_i \in X, v_i \in D_{x_i}\}$,
- $E = \{\ \{(x_i, v_i), (x_j, v_j)\}\ \mid\ i \neq j, C_{ij} \notin C \text{ or } C_{ij} \in C, (v_i, v_j) \in R(C_{ij})\}$

The micro-structure was introduced by Jégou in order to detect new tractable classes for CSP based on graph theory. In [2], Cohen showed that the class of binary CSPs with triangulated complement of micro-structure is tractable and arc-consistency is a decision procedure.

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After that, Jégou and al. in [8] presented new results on the effectiveness of classical algorithms when the number of maximal cliques in graph of micro-structure can be bounded by a polynomial.

Therefore, the goal of this report is to generalize the definition of micro-structure to non-binary CSPs.

2 Generalized Micro-Structures

The first extension of the notion of micro-structure to non-binary CSPs was proposed by Cohen in [2]: this generalization is based on hypergraphs [3]. In contrast, our generalisations are inspired by technical methods of conversion between non-binary and binary CSPs [12, 1]. Then, our micro-structure are a simple undirected graph obtained by using binary encoding based on graphic representation of non-binary CSPs: the well known methods are the dual encoding (also called dual representation), hidden transformation (also called hidden variable representation) and mixed encoding.

2.1 Generalized micro-structure based on dual representation

The dual encoding was introduced by Dechter and Pearl in [4] and is based on the dual graph representation (also called intergraph in [5, 7] or line graph in graph theory) which comes from the relational database. In this encoding, the constraints of the original problem become variables (also called dual variables). The domain of each new variable is exactly the set of tuples allowed by the original constraint. We define a constraint between two dual variables if the original constraints share at least one variable.

Definition 2 (generalized micro-structure based on dual representation) *Given a CSP $P = (X, D, C)$ (not necessarily binary), the* generalized micro-structure *of P is the undirected graph $\mu_{G_d}(P) = (V, E)$ with:*

- $V = \{(C_i, t_i) : C_i \in C, t_i \in R(C_i)\}$,
- $E = \{ \ \{(C_i, t_i), (C_j, t_j)\} \ \mid \ i \neq j, t_i[S(C_i) \cap S(C_j)] = t_j[S(C_i) \cap S(C_j)]\}$

As for the micro-structure, there is a direct relationship between cliques and solutions of CSPs:

Theorem 1 *A CSP P has a solution iff $\mu_{G_d}(P)$ has a clique of size e.*

Proof: By construction, $\mu_{G_d}(P)$ is e-partite, and any clique contains at most one vertex (C_i, t_i) per constraint $C_i \in C$. Hence the e-cliques of $\mu_{G_d}(P)$ correspond exactly to its cliques with one vertex (C_i, t_i) per constraint $C_i \in C$. Now by construction of $\mu_{G_d}(P)$ again, any two vertices (C_i, t_i), (C_j, t_j) joined by an edge (in particular, in some clique) satisfy $t_i[S(C_i) \cap S(C_j)] = t_j[S(C_i) \cap S(C_j)]$. Hence all t_i's in a clique join together, and it follows that the e-cliques of $\mu_{G_d}(P)$ correspond exactly to tuples t which are joins of one allowed tuple per constraint, that is, to solutions of P. □

The example below will be used throughout the paper:

Example 1 *The figure 1 presents a CSP $P = (X, D, C)$ with :*
$X = \{x_1, x_2, x_3, x_4, x_5\}$,
$D = (D_{x_1}, D_{x_2}, D_{x_3}, D_{x_4}, D_{x_5})$ *with*

$D_{x_1} = \{a, a'\}$, $D_{x_2} = \{b\}$, $D_{x_3} = \{c\}$, $D_{x_4} = \{d, d'\}$ *and* $D_{x_5} = \{e\}$.
$C = \{C_1, C_2, C_3, C_4\}$ *is a set of four constraints with* $S(C_1) = \{x_1, x_2\}$, $S(C_2) = \{x_2, x_3, x_5\}$, $S(C_3) = \{x_3, x_4, x_5\}$ *and* $S(C_4) = \{x_2, x_5\}$.
The relations associated to the previous constraints are given by these tables:

$R(C_1)$	
x_1	x_2
a	b
a'	b

$R(C_2)$		
x_2	x_3	x_5
b	c	e

$R(C_3)$		
x_3	x_4	x_5
c	d	e
c	d'	e

$R(C_4)$	
x_2	x_5
b	e

The generalised micro-structure based on dual encoding of the last example is shown in figure 1.

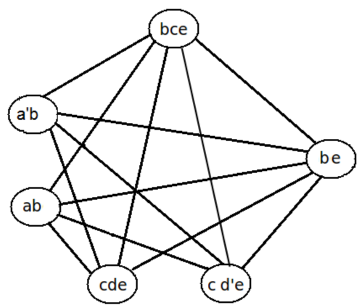

Fig. 1. Generalized micro-structure built by using dual representation.

We have four constraints, then $e = 4$. Thanks to Theorem 1, a solution of P is a clique of size 4, e.g. (ab, bce, be, cde).

The generalized micro-structure based on minimal intergraph [5, 7] is similar to generalized micro-structure based on dual encoding because the deleted edges in the minimal intergraph will be added by the relation $t_i[S(C_i) \cap S(C_j)] = t_j[S(C_i) \cap S(C_j)]$.

2.2 Generalized micro-structure based on hidden transformation

The hidden variable encoding was inspired by Peirce [9] (cited in [11]). In hidden transformation, the set of variables contains the original variables plus the set of dual variables. There is a binary constraint between a dual variable and original variable if the original variable belongs to the scope of dual variable [12].

Definition 3 (generalized micro-structure based on hidden transformation) *Given a CSP* $P = (X, D, C)$*, the* generalized micro-structure based on hidden transformation *of* P *is the undirected graph* $\mu_{G_h}(P) = (V, E)$ *with:*

- $V = S_1 \cup S_2$
- $S_1 = \{(C_i, t_i) : C_i \in C, t_i \in R(C_i)\}$,
- $S_2 = \{(x_j, v_j) : x_j \in X, v_j \in D_{x_j}\}$,
- $E = \{\ \{(C_i, t_i), (x_j, v_j)\}\ \mid$ *either* $x_j \in S(C_i)$ *and* $v_j = t_i[x_j]$ *or* $x_j \notin S(C_i)\}$

We now turn to relationship between bicliques and solutions of CSPs. Before that, we should recall that a biclique is a complete bipartite graph, i.e. a bipartite graph in which every vertex of the first set is connected to all vertices of the second set.

Proposition 1 *In a generalized micro-structure, a $K_{n,e}$ biclique with e tuples, such that no two tuples belong to the same constraint, cannot contain two different values of the same variable.*

Proof: We assume that a $K_{n,e}$ biclique with e tuples, such that no two tuples belong to the same constraint, can contain two different values v_j and v'_j of the same variable x_j. Therefore, there is at least a constraint C_i such that $x_j \in S(C_i)$. Thus, $t_i[x_j] = v_j$, v'_j or another $v"_j$ and in all three cases, the assumption is false because t_i cannot be connected to two different values of the same variable. □

Proposition 2 *In a generalized micro-structure, a $K_{n,e}$ biclique with n values, such that no two values belong to the same domain, cannot contain two different tuples of the same constraint.*

Proof: We assume that a $K_{n,e}$ biclique with n values, such that no two values belong to the same variable, can contain two different tuples t_i and t'_i of the same constraint C_i. Therefore, there is at least a variable x_j such that $t_i[x_j] \neq t'_i[x_j]$. If $t_i[x_j] = v_j$ and $t'_i[x_j] = v'_j$ must belong to the same biclique. Thus, the assumption is false because we cannot have two values of a same variable.□

Using these two properties, we can deduce the following theorem :

Theorem 2 *Given a CSP $P = (X, D, C)$ and $\mu_{G_h}(P)$ its generalized micro-structure, P has a solution iff $\mu_{G_h}(P)$ has a $K_{n,e}$ biclique with n values and e tuples such that no two values belong to the same domain and no two tuples belong to the same constraint.*

Proof: A $K_{n,e}$ biclique with n values and e tuples such that no two values belong to the same domain and no two tuples belong to the same constraint correpond to a consistent assignment which satisfies all constraints. □

Figure 2 represents the generalized micro-structure based on hidden transformation of the CSP of example 1.

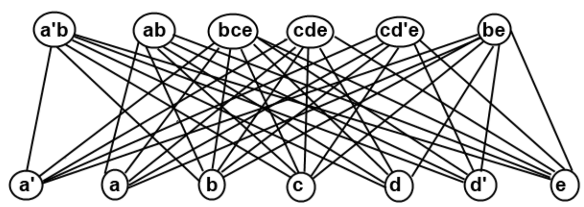

Fig. 2. Generalized micro-structure built by using hidden variable.

Based on the previous example, we can easily see that a biclique does not necessarily correspond to a solution. Although, $\{a, a', b, c, e, ab, ab', bce, be\}$ is $K_{5,4}$ biclique but it is not a solution. Contrary to $\{a, b, c, d, e, ab, bce, be, cde\}$ that is a $K_{5,4}$ biclique and is a solution of P.

Then, the set of solutions is not equivalent to the set of bicliques. This is due to the manner which the graph of micro-structure is completed. For the next generalization, we will propose another manner to complete the graph of micro-structure: this new way of representation can also be deduced from hidden encoding.

2.3 Generalized micro-structure based on mixed encoding

We finally turn to the last interesting method of translation which is called mixed encoding. It allows us to connect the dual variables to the original variables, two tuples of two different constraints and two values of two different variables.

In other words, it uses at the same time dual encoding and hidden variable encoding. More description is given by the following definition.

Definition 4 (generalized micro-structure based on mixed encoding) *Given a CSP* $P = (X, D, C)$ *, the* generalized micro-structure based on mixed encoding *of* P *is the undirected graph* $\mu_{G_m}(P) = (V, E)$ *with:*

- $V = S_1 \cup S_2$
- $S_1 = \{(C_i, t_i) : C_i \in C, t_i \in R(C_i)\}$,
- $S_2 = \{(x_j, v_j) : x_j \in X, v_j \in D_{x_j}\}$,
- $E = E_1 \cup E_2 \cup E_3$
- $E_1 = \{\ \{(C_i, t_i), (C_j, t_j)\}\ \mid\ i \neq j, t_i[S(C_i) \cap S(C_j)] = t_j[S(C_i) \cap S(C_j)]\}$
- $E_2 = \{\ \{(C_i, t_i), (x_j, v_j)\}\ \mid\ \textit{either}\ x_j \in S(C_i)\ \textit{and}\ v_j = t_i[x_j]\ \textit{or}\ x_j \notin S(C_i)\}$
- $E_3 = \{\ \{(x_i, v_i),\ (x_j, v_j)\}\mid\ i \neq j\ \textit{and}\ x_i \neq x_j\}$

The generalized micro-structure based on mixed encoding of the CSP of example 1 is shown in figure 3.

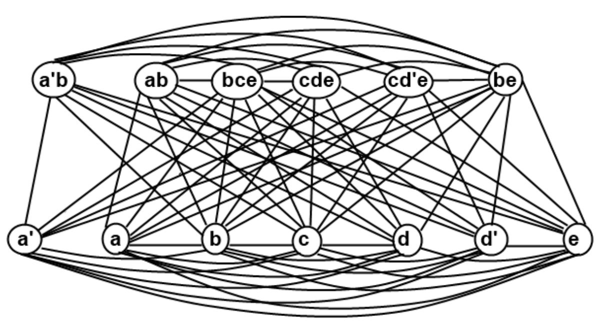

Fig. 3. Generalized micro-structure built by using mixed encoding.

Proposition 3 *In a generalized micro-structure, a* K_{n+e} *clique cannot contain more than n values where each pair of values derived from two different variables.*

Proof: We assume that proposition 3 is false, then a K_{n+e} clique must contain two values v_i and v'_i of the same variable x_j. Thus, the corresponding vertices to v_i and v'_i cannot be adjacent and cannot belong to the same clique. □

Proposition 4 *In a generalized micro-structure, a* K_{n+e} *clique cannot contain more than e tuples where each couple of tuples derived from two different constraint.*

Proof: (similar to the previous proof)

According to the last properties, there is a strong relationship between cliques and solutions of CSPs:

Theorem 3 *A CSP P has a solution iff $\mu_{G_m}(P)$ has a clique of size $n+e$.*

Proof: In a generalized micro-structure based on mixed encoding, a K_{n+e} clique correspond to a consistent assignment of n variables which satisfies e constraints (based on the two previous propositions). Then, it is a solution to P. □

The first micro-structure is exactly the micro-structure of the dual CSP, but the second and the last representations correspond neither to the micro-structure of hidden CSP nor to the micro-structure of a mixed CSP because of the way to complete these two graphs.

3 Conclusion

We have investigated the different binary representation of n-ary CSPs in order to define a generalization of micro-structure for non-binary CSPs. The first perspective is to check whether the results in [8] can be extended to non-binary CSPs represented by one of this micro-structure. The second aims to provide formal frameworks for studying properties on the complexity of n-ary CSPs as with the binary case.

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